Chapter 9
Dynamic Models of Investment

In this chapter we present the main neoclassical model of investment, under convex adjustment costs. This model is known as the “q model of investment”, where q is the ratio of the market value of installed capital relative to its replacement value. We discuss the model both under conditions of certainty and under uncertainty.

So far we have been assuming that firms choose their capital stock so that the marginal product of capital equals the user cost of capital, as determined by the real interest rate and the rate of depreciation. This theory in fact determines the optimal capital stock and not the amount of investment. Investment flows determine how quickly a firm moves from its current to the optimal capital stock. When firms can adjust their capital stock immediately and without cost, the flow of investment is not defined, as the capital stock jumps immediately to its optimal level.

In fact, however, the change of the capital stock involves adjustment costs. A firm that chooses to raise its stock of productive capital should rent or buy additional space, buy and install new equipment, and train employees to use the extra equipment. In additional there are delivery lags and installation costs, making it more costly to adjust the capital stock quickly. All these costs are beyond the cost of buying additional capital goods. In addition, it is to be expected that these adjustment costs will be convex, i.e. they will depend on the size of the new investment. The higher the size of new investment, the greater will be the average adjustment cost of installing (or de-installing) an additional unit of capital.

In the presence of adjustment costs, the investment decisions of firms will thus not only depend on present conditions, such as the relation of the user cost of capital to the marginal product of capital, but also on past and expected future decisions. The problem of the firm becomes truly dynamic. Jorgenson (1963) assumed that, precisely because of the existence of adjustment costs, firms are not immediately but only gradually adjusting their stock of capital towards its “optimal” level, as determined by the user cost of capital and the marginal product of capital. He thus postulated an investment function which determined current investment as a fraction of the difference between the current and the “optimal” (desired) capital stock.

However, Jorgenson did not derive the speed of adjustment, and thus the flow of investment, from a fully dynamic optimization problem. This was accomplished later by Lucas (1967), Gould (1968) and Treadway (1969), who, instead of postulating the investment function, as Jorgensen had done, solved for the optimal investment function from the dynamic problem of a firm maximizing the present value of its profits, subject to convex costs of adjusting its capital stock. Soon afterwards,
Lucas and Prescott (1971) extended this framework to examine the determination of investment under uncertainty.\(^1\)

A second approach to the problem of investment was that of Tobin (1969), who compared the ratio of the market value of installed capital of a firm, to the replacement cost of capital, naming this ratio \(q\). Tobin argued that if the already installed capital stock of a firm has higher value than the cost of replacing the capital goods that compose it, i.e. if \(q\) is greater than one, then it will be profitable for the firm to invest, i.e. purchase and install new capital goods. Tobin argued that the rate of investment will be an increasing function of \(q\). The ratio of the value of the already installed capital stock to its replacement cost has since been called “Tobin’s \(q\)”. However, much like Jorgenson, Tobin did not derive his investment function from a dynamic optimization problem either.

Several years later, Abel (1982) and Hayashi (1982) showed that Tobin’s “\(q\) theory” and the theory of “adjustment costs” for investment of Jorgenson, as modeled by Lucas, Prescott, Gould and Treadway, can be combined in a unified framework. This synthesis of the two theories is now considered as the main neoclassical dynamic model of investment.

The firm does not choose the level of its capital stock, by equating at any time the marginal product of capital to the sum of the real interest rate and the depreciation rate, but it chooses the amount of investment, taking into account the adjustment costs of the capital stock. Since marginal adjustment costs increase with the amount of investment, investment results in a gradual adjustment of the capital stock towards its steady state value. On the adjustment path, the firm takes into account both the current and future effects of its investment decisions. Thus, investment depends on both current and expected future developments in the value of the marginal product of capital and the user cost of capital.

**9.1 Optimal Investment with Convex Adjustment Costs**

We consider a competitive firm producing a good \(Y\). The production function of the firm is given by,

\[
Y(t) = AF(K(t))
\]

(9.1)

where \(A\) is total factor productivity and \(K\) the capital stock. The production function is characterized by diminishing returns. The market price of output and the capital stock is equal to unity.\(^2\)

In order to change its capital stock, the firm must undertake gross investment \(I\). The change in its capital stock is thus determined by,

\[
I(t) = \dot{K}(t) + \delta K(t)
\]

(9.2)

\(^1\) These models of investment are sometimes referred to as “flexible accelerator” models of investment, in contrast to earlier “fixed accelerator” models, which modeled investment as a constant multiple of the change in output or consumption. For a survey of these earlier “accelerator” models see Knox (1952). The most famous application of the earlier accelerator models in economics has been the paper of Samuelson (1939), which combined the multiplier and the accelerator to characterize output fluctuations in a keynesian model. We shall analyze the Samuelson model in Chapter 12.

\(^2\) To simplify the analysis we assume that capital is the only factor of production. The nature of the results does not depend on this simplifying assumption.
where $\delta > 0$ is a constant depreciation rate.

We assume that the cost of gross investment for the firm is equal to,

$$I(t) + \psi(I(t))$$

(9.3)

where $\psi$ is a convex function, for which it holds that $\psi(0) = 0$, $\psi' > 0$ και $\psi'' \geq 0$.

The convex function $\psi$ measures the installation (adjustment) cost of gross investment. It is assumed that the higher the size of gross investment, the higher the marginal installation cost. The total cost of gross investment is thus equal to the cost of buying the relevant capital goods, plus the installation (adjustment) cost.

Figure 9.1 depicts the installation cost as a function of the size of gross investment of the firm. The installation cost rises as gross investment rises. If the second derivative is positive, the installation cost rises at a rising rate. The adjustment cost function is assumed symmetric. Thus, what applies to positive investment also applies to negative investment.

The instantaneous profits of the firm are thus equal to,

$$\Pi(t) = Y(t) - I(t) - \psi(I(t))$$

(9.4)

9.1.1 The Choice of Optimal Investment

Assume that a time $t_0$ the firm chooses an investment path that maximizes the present value of current and future profits. Assuming an infinite time horizon, the present value of the profits of the firm is equal to,

$$V(0) = \int_{t_0}^{\infty} e^{-rt} (Y(t) - I(t) - \psi(I(t))) dt$$

(9.5)

where $r$ is the real interest rate, assumed exogenous and constant.

The present value (9.5) is maximized under the constraint of the production function (9.1) and the investment function (9.2), which links gross investment to the accumulation of capital.

The current value Hamiltonian of this problem is defined by,

$$\left(AF(K(t)) - I(t) - \psi(I(t))\right) + q(t)(I(t) - \delta K(t))$$

(9.6)

where $q(t)$ is the multiplier of the capital accumulation constraint. $q(t)$ is the shadow value of an additional unit of capital at $t$.

From the first order conditions for a maximum, it follows that,
The interpretation of this form of the first order conditions is straightforward.

Condition (9.7) determines the shadow value of an additional unit of capital \( q \) as equal to the marginal cost of investment. This is equal to the purchase price of capital goods (assumed equal to unity), plus the marginal installation cost \( \psi'(I(t)) \).

Condition (9.8) requires that the firm will invest until the user cost of capital (on the left hand side) is equal to the marginal product of capital (on the right hand side). The user cost of capital is the real interest rate, plus the depreciation rate, minus the expected appreciation rate of the capital stock, multiplied by the shadow value of capital.

9.1.2 The Case of Zero Adjustment Costs

In the case of zero adjustment costs, the marginal adjustment cost of the capital stock \( \psi' \) is equal to zero. In this case, conditions (9.7) and (9.8) imply,

\[
q(t) = 1 + \psi'(I(t)) = 1 + \psi'(\dot{K}(t) + \delta K(t)) \tag{9.7'}
\]

\[
\left( r + \delta - \frac{\dot{q}(t)}{q(t)} \right) q(t) = A \frac{\partial F(K(t))}{\partial K(t)} = AF_K(K(t)) \tag{9.8'}
\]

These are the usual first order conditions we have utilized so far. The variables \( q \) and \( K \) jump immediately to their equilibrium values. The shadow value of capital is continuously equal to unity, i.e. the purchase price of capital goods, and the capital stock adjusts immediately to the level where the marginal product of capital is equal to the real interest rate plus the depreciation rate, \( r + \delta \). There is no investment flow, as the capital stock adjusts immediately. Without adjustment costs, this model does not determine gross investment, but only the equilibrium capital stock.

9.1.3 The Investment Function with Convex Adjustment Costs

Let us now return to the general case, where there is a strictly convex adjustment cost function for the capital stock.

From (9.7) it follows that investment is a positive function of \( q - 1 \). Solving (9.7) for \( I \) we get that,

\[
I(t) = (\psi')^{-1}(q(t) - 1) \tag{9.9}
\]

Gross investment depends only on the difference of the shadow price of installed capital \( q \) from unity, as assumed by Tobin. This dependence is positive because the marginal cost of investment is
positive. For this reason, the theory that depends on a rising adjustment cost on investment is referred to as the \( q \) theory of investment.\(^3\)

9.1.4 The Determinants of \( q \)

We have already defined \( q \) as the shadow price of an additional unit of installed capital. We next turn to the determinants of \( q \).

From the first order condition (9.7), \( q \) is equal to the marginal cost of investment. We have already used this condition to derive the investment function (9.9).

To analyze the determinants of \( q \), we must look into the second first order condition (9.8). This can be re-written as,

\[
q(t) = (r + \delta)q(t) - AF_{K}(K(t))
\]

(9.10) is a first order linear differential equation with variable coefficients. Its solution takes the form,

\[
q(t) = \int_{t_0}^{\infty} e^{-(r+\delta)(s-t)} AF_{K}(K(s)) ds
\]

(9.11)

From (9.11) it follows that \( q \) is the present value of all future marginal products of capital. As a result, \( q \) depends negatively on the real interest rate and the depreciation rate, as well as factors that reduce the marginal product of capital, such as the capital stock. \( q \) depends positively on factors that increase the marginal product of capital, such as total factor productivity.\(^4\)

In equilibrium, \( q \) is determined so that both first order conditions are satisfied. Thus, in equilibrium the marginal cost of investment is equal to the expected present value of future marginal products of capital. This is the condition that determines optimal investment.

9.1.5 The Dynamic Adjustment of \( q \) and the Capital Stock \( K \).

The determination of the shadow price of an additional unit of capital \( q \), and the stock of capital \( K \), can be inferred from the differential equations (9.7) and (9.8). Since both of these differential equations are nonlinear, their solution can be described by a phase diagram, as in Figure 9.2.

For a constant capital stock, (9.7) implies that,

\[
q = 1 + \psi'(\delta K)
\]

(9.12)

The higher the stock of capital, the higher will be the shadow value \( q \) implied by (9.12), as depreciation investment will be higher. (9.12) is depicted as the curve with a positive slope in

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\(^3\) It is worth noting that investment depends on the so-called marginal \( q \). For the relationship between the marginal and the average \( q \) see Hayashi (1982).

\(^4\) Both Abel (1982) and Hayashi (1982) also analyzed the impact of the tax treatment of investment on \( q \) and gross investment.
Figure 9.2, and can be interpreted as the constant capital stock condition. When \( q \) is higher than the level implied by (9.12), the capital stock increases, as gross investment is higher than depreciation investment. The opposite happens when \( q \) is lower than the level implied by (9.12). Then the capital stock decreases.

For a constant \( q \), (9.8) implies that

\[
q = \frac{1}{r + \delta} AF_n(K)
\]

(9.13)

The higher the stock of capital, the lower the shadow value of capital \( q \), as the marginal product of capital is a negative function of the stock of capital. (9.13), the curve with a negative slope in Figure 9.2, can be seen as the constant gross investment condition. When \( K \), the stock of capital stock is higher than implied by (9.13), the marginal product of capital is lower than what would be required by (9.13), and \( q \) increases, so that the user cost of capital is also lower. Thus, investment is low but increasing over time. The opposite happens when the capital stock is lower than the level implied by (9.13).

The steady state is determined at the intersection of the two curves (9.12) and (9.13). This steady state equilibrium is a saddle point, since \( q \) is a non predetermined variable and \( K \) is a predetermined variable. The adjustment path is unique and is depicted in Figure 9.2.

In Figure 9.3 we analyze the effects of a permanent increase in the real interest rate. This leads to increase in the user cost of capital and a downward shift of the constant investment locus. Since the capital stock is predetermined in the short run, this leads to an immediate reduction of \( q \) and investment, and a gradual reduction of the stock of capital. As the capital stock decreases, the marginal product of capital gradually increases, and this leads to a gradual increase in both \( q \) and investment. In the new equilibrium \( E' \), both the stock of the principal and \( q \) are at a lower level than the initial equilibrium \( E \).

In Figure 9.4 we analyze the effects of a permanent increase in total factor productivity \( A \). This leads to an increase in the marginal product of capital and a shift of the equilibrium investment locus upwards. Since the capital stock is predetermined in the short run, this leads to an immediate increase of \( q \) and investment, and a gradual increase of the stock of capital. As the capital stock increases, the marginal product of capital gradually falls, and this leads to a gradual reduction of both \( q \) and investment. In the new equilibrium \( E' \), both the stock of capital and \( q \) (investment) are at a higher level than the initial equilibrium \( E \).

This is the basic neoclassical model of investment with convex adjustment costs of investment. This model can be generalized so that the adjustment cost function depends not only on gross investment, but also on the stock of capital. It can also be generalized to simultaneously analyze investment and labor demand. Finally, it can also be generalized to allow for product market imperfections, as well as to the case of uncertainly.

9.2 Optimal Investment under Uncertainty

We now turn to the case of investment under uncertainty. As in the case of consumption, uncertainty introduces additional problems.
Just as under certainty, we shall assume that the firm chooses its investment path in order to maximize its value to its owners. The value is equal to the present value of the profits that the firm generates.

9.2.1 The Value of a Firm under Uncertainty

Whereas under certainty the problem of the maximization of the present value of the profits of the firm is easily defined, under uncertainty, the question that arises is what should be the discount rate at which firms should discount future profits.

Let us assume that $V_t$ is the value of the firm in period $t$, and $\Pi_t$ is its per period revenue, net of investment expenditures. Then the rate of return $1+\pi_t$ from holding the firm for one period, will be given by,

$$1 + \pi_t = \frac{V_{t+1} + \Pi_{t+1}}{V_t}$$ \hspace{1cm} (9.14)

For a consumer that invests in the firm under uncertainty, the rate of return from holding the firm $1+\pi_t$, and hence $V_t$ and $\Pi_t$ must satisfy,

$$u'(C_t) = \frac{1}{1+\rho} E_t \left[ (1 + \pi_t) u'(C_{t+1}) \right] = \frac{1}{1+\rho} E_t \left[ \left( \frac{V_{t+1} + \Pi_{t+1}}{V_t} \right) u'(C_{t+1}) \right]$$ \hspace{1cm} (9.15)

(9.15) is the same as the first order condition (7.12) for investing in a “risky” asset, in the problem analyzed in Chapter 8. The returns generated by the firm in each state of nature are weighted by the marginal utility of consumption in that state. The discount factor that must be applied must take into account the correlation of the firm’s profits with the marginal utility of consumption at each state.

Solving the first order condition (9.15) recursively forward, assuming away bubbles, we get the “fundamental solution” for the value of the firm $V_t$ as,

$$V_t = E_t \left( \sum_{s=1}^{\infty} \left( \frac{1}{1+\rho} \right)^{s} \frac{u'(C_{t+s})}{u'(C_t)} \Pi_{t+s} \right)$$ \hspace{1cm} (9.16)

This shows that the value of the firm is equal to the present discounted value of expected future profits. The discount rate for each period and for each state of nature is the marginal rate of substitution between consumption at time $t$ and consumption at that period and that state of nature. This has the implication that the higher the correlation between a firm’s profits and consumption, the higher will be the discount factor applied, and the lower the value of the firm.

In practice, it is often assumed that firms maximize the present discounted value of profits by using a deterministic discount rate. For firms opting to do this, (9.16) becomes,

$$V_t = E_t \left( \sum_{s=1}^{\infty} \left( \Pi_{t+s} \frac{1}{1+r_{t+s}} \right) \right)$$ \hspace{1cm} (9.17)
where \( r_{t+z} \) is a deterministic interest rate in period \( t+z \). Although widely used, a specification such as (9.17) is generally inappropriate, because it suggests that at each date, the same discount factor is used to evaluate returns in different states of nature.

An equation such as (9.17) can be justified only under very specific assumptions.

One set of assumptions that can be used to justify it is the assumption of risk neutrality on the part of consumers. If consumers are risk neutral, so their utility is linear in consumption and their marginal utility of consumption is constant, then the discount rate is not only deterministic, but also constant, and equal to the pure rate of time preference \( \rho \). Thus, in the case of risk neutrality (9.17) simplifies further to,

\[
V_t = E_t \left( \sum_{s=1}^{\infty} \frac{1}{1+\rho} \Pi_{t+s} \right)
\]

(9.18)

Another set of assumptions that can be used to justify a deterministic discount rate is to assume that investment decisions do not affect the relative distribution of returns across states of nature, but only the scale of the firm. In this case the firm can use a constant discount rate, equal to the risk free rate, plus a risk premium that reflects the specific risk associated with the firm’s activities.

Both sets of assumptions are unlikely to hold in general, but they are often used as convenient approximations. It is worth noting however that they are good approximations only when considerations of risk aversion are not central to the problem analyzed.

9.2.2 The Lucas and Prescott Model of Investment under Uncertainty

We next turn to an examination of the investment decisions of a competitive firm under uncertainty, assuming that the objective of the firm is to maximize value as defined in (9.18), with a deterministic discount rate. The model we analyze is a linear quadratic variant of the class of models introduced by Lucas and Prescott (1971), and is similar in many respects to the \( q \) model we analyzed in section 9.1.

We assume a competitive firm \( i \) that takes market prices as given. Its profit in period \( t \) is defined by,

\[
\Pi_t = \left( p_t Y_t - I_t - \frac{\psi}{2} (I_t)^2 \right)
\]

(9.19)

where \( p \) is the competitive relative price of its output \( Y \), and \( I \) is gross investment. \( \psi \) is a constant positive parameter measuring the strength of investment adjustment costs, which are quadratic in gross investment. The price of investment goods is normalized to unity.

Output is produced using capital, through a linear production function of the form,

\[
Y_t = AK_t
\]

(9.20)

where \( A \) is total factor productivity, assumed to constant and the same for all firms.
The relation between gross investment and the evolution of the capital stock is determined by,

\[ K_{t+1} = I_t + (1 - \delta)K_t \]  

(9.21)

where \( \delta \) is the constant depreciation rate.

\( i \) is continuous in the interval [0,1]. Thus, industry output is given by,

\[ Y_i = \int_{i=0}^{1} Y_idi \]  

(9.22)

Since all firms face the same technology and the same market prices, the output of all firms will be the same. Thus, from now on we treat \( Y \) as the output of the representative firm. The same goes for all other variables, such as investment and the capital stock.

Under the assumptions we have made, the representative firm maximizes its present value,

\[ V_t = E_t \left( \sum_{s=1}^{\infty} \frac{1}{1+r} \Pi_{t+s} \right) = E_t \left( \sum_{s=1}^{\infty} \frac{1}{1+r} \left( p_{t+s}AK_{t+s} - I_{t+s} - \frac{\psi}{2}(I_{t+s})^2 \right) \right) \]  

(9.23)

subject to the sequence of accumulation equations (9.21). The stochastic processes driving the market price, the relative price of capital goods and productivity are taken as given.

The Lagrangian of this problem is defined by,

\[ E_t \left( \sum_{s=1}^{\infty} \frac{1}{1+r} \left( p_{t+s}AK_{t+s} - I_{t+s} - \frac{\psi}{2}(I_{t+s})^2 \right) + q_{t+s}(I_{t+s} + (1-\delta)K_{t+s} - K_{t+s+1}) \right) \]  

(9.24)

where \( q_{t+s} \) is the sequence of Lagrange multipliers of the capital accumulation constraints.

From the first order conditions for a maximum we get the two familiar first order conditions,

\[ q_t = 1 + \psi I_t \]  

(9.25)

\[ (1+r)q_t - (1-\delta)E_tq_{t+1} = E_t\rho_{t+1}A \]  

(9.26)

(9.25) and (9.26) have interpretations similar to the interpretation of the corresponding conditions (9.7) and (9.8) in the deterministic case analyzed in the previous section.

(9.25) requires that at the optimum the firm equates the shadow value of an addition to its capital stock \( q \) to the marginal cost of investment. The latter consists of the price of purchasing capital goods, assumed to be equal to one, plus the adjustment cost of investment.
(9.26) requires that the user cost of capital, as measured by the term on the left hand side, is equal to the expected future value of the marginal product of capital, as measured by the term on the right hand side.

Solving (9.26) forward for \( q \), we get,

\[
q_t = \frac{1}{1+r} E_t \sum_{s=0}^{\infty} \left( \frac{1-\delta}{1+r} \right)^s p_{t+s+1} A
\]

(9.27)

\( q \) turns out to be the discounted value of all expected future values of the marginal product of capital. It depends positively on the expected future evolution of the relative price for the product of the firm and the marginal productivity of capital \( A \). It also depends negatively on the discount rate \( r \) and the depreciation rate \( \delta \).

To determine investment, we can solve (9.25) for investment. This results in,

\[
I_t = \frac{1}{\psi} (q_t - 1)
\]

(9.28)

From (9.28), investment depends positively on the difference of \( q \) from unity, which is the purchase price of investment goods. Substituting (9.27) in (9.28), investment of the representative firm will be determined by,

\[
I_t = \frac{1}{\psi} \left( \frac{1}{1+r} \left( E_t \sum_{s=0}^{\infty} \left( \frac{1-\delta}{1+r} \right)^s p_{t+s+1} A \right) - 1 \right)
\]

(9.29)

Investment thus depends positively on the discounted value of all expected future changes in the value of the marginal product of capital, and negatively on the depreciation rate and the adjustment cost parameter \( \psi \).

From (9.29), the capital stock of the representative firm evolves according to,

\[
K_{t+1} = (1-\delta)K_t + \frac{1}{\psi} \left( \frac{1}{1+r} \left( E_t \sum_{s=0}^{\infty} \left( \frac{1-\delta}{1+r} \right)^s p_{t+s+1} A \right) - 1 \right)
\]

(9.30)

In order to say more about the determination of investment and the capital stock, one must make specific assumptions about the stochastic process driving the market price of output.

Although for each competitive firm the price of output is taken as given, for the industry, the market price will be determined endogenously, from the equation of total demand for its product and industry supply. Industry supply will depend on investment and the evolution of the capital stock. This is something that Lucas and Prescott (1971) took explicitly into account, solving for the equilibrium price endogenously, as a function of the capital stock, and characterizing the evolution of the capital stock and the equilibrium price in a rational expectations equilibrium.

Assume that industry demand is linear in the price and given by,
where $D>0$, $b>0$ are constant parameters. $D$ measures the size of the market, and $b$ the price responsiveness of demand. $v$ is a stochastic disturbance to industry demand.

From (9.31), and after substituting for output from the production function, the competitive price is determined as,

$$p_t = \frac{1}{b} (D - Y_t + v_t) = \frac{1}{b} (D - AK_t + v_t)$$  \hspace{1cm} (9.32)

We can use (9.32) to substitute for the expected equilibrium price in the capital accumulation equation (9.30), and solve for the evolution of the capital stock as a function of only exogenous shocks.

Note that using the forward shift operator, $Fp_t = E_t (p_{t+1})$, (9.30) can be written as,

$$K_{t+1} = (1 - \delta)K_t + \frac{1}{\psi} \left\{ \frac{1}{(1+r)} \left[ E_t \sum_{s=0}^{\infty} \left( \frac{1 - \delta}{1 + r} \right)^s F^{s+1} p_{t+s} A \right] - 1 \right\} = (1 - \delta)K_t + \frac{1}{\psi} \left\{ \frac{AFp_t}{(1+r) - (1-\delta)F} - 1 \right\}$$  \hspace{1cm} (9.33)

Multiplying both sides of (9.33) by $(1+r)-(1-\delta)F$ and collecting the terms containing $K$ on the left hand side, and using (9.32) to substitute out for $p_t$, we get,

$$\left( F^2 - \frac{(1+r)+(1-\delta)^2+(A^2/\psi b)}{(1-\delta)} F + (1+r) \right) K_t = -\frac{1}{\psi(1-\delta)} \left\{ A \left( D + F v_t \right) - (r+\delta) \right\}$$  \hspace{1cm} (9.34)

The characteristic polynomial of the quadratic equation involving $F$ on the left hand side has two roots that lie on either side of unity. We can thus rewrite (9.34) as,

$$(F - \lambda)(F - \mu) K_t = -\frac{1}{\psi(1-\delta)} \left\{ A \left( D + F v_t \right) - (r+\delta) \right\}$$  \hspace{1cm} (9.35)

where $\lambda < 1$ is the smaller root, and $\mu > 1$ the larger root. It follows from (9.34) that $\lambda = (1+r)/\mu$, as the roots satisfy,

$$\lambda + \mu = \frac{(1+r)+(1-\delta)^2+(A^2/\psi b)}{(1-\delta)} > 2 \text{ and } \lambda \mu = 1 + r$$

From (9.35), the capital stock follows,
From (9.36), the evolution of the industry capital stock in rational expectations equilibrium depends on current expectations about the whole future path of disturbances to industry demand, and parameters such as the discount rate $r$, the productivity of capital $A$, the adjustment cost parameter $\psi$, the depreciation rate $\delta$, the size of the market $D$ and the price responsiveness of industry demand $b$. $\lambda$ depends only on the discount rate and technological parameters.

To get a closed form solution, we must make assumptions about the exogenous stochastic process driving industry demand. Let us assume that $v$ follows a stationary AR(1) process of the form,

$$v_t = \theta v_{t-1} + \varepsilon_t$$

where $0 < \theta < 1$ and $\varepsilon$ is a white noise process.

Under this assumption, (9.36) implies that,

$$K_{t+1} = \lambda K_t + \frac{\lambda}{\psi(1-\delta)(1+r-\lambda)} \left( \frac{AD}{b} - (r + \delta) \right) + \frac{\lambda}{b \psi(1-\delta)(1+r)} E_T \sum_{s=0}^{\infty} \left( \frac{\lambda}{1+r} \right)^s v_{t+s+1}$$

Investment and the evolution of the capital stock depend only on the current shock to aggregate demand, because the current shock is a sufficient statistic for all future shocks to industry demand.

Thus, the structure of the full Lucas and Prescott model is as follows. The representative firm chooses investment, and implicitly the capital stock and output, to maximize the present value of its profits, taking as given the market price of its output, the exogenous relative price of capital goods and exogenous productivity $A$. In order to compute the full equilibrium, once investment and output of the representative firm are determined, industry output is replaced in the industry demand function to solve for the equilibrium price in terms of the exogenous stochastic process driving industry demand. The full equilibrium is thus described by a pair of interrelated capital accumulation and price equations, which are consistent with continuous market clearing.\(^5\)

This model can be generalized in a number of directions. First, one could introduce additional shocks, such as shocks to total factor productivity. Second, one could introduce labor in the production function, and study interactions between investment and labor demand. Third, one could introduce business taxes. Fourth, one could introduce externalities from capital accumulation. Finally, one could assume imperfect rather than perfect competition.

### 9.3 Conclusions

In this chapter we have presented the main neoclassical model of investment, under convex adjustment costs. This model is known as the “$q$ model of investment", where $q$ is the ratio of the market value of installed capital relative to its replacement value. We analyzed the model both under conditions of certainty and under uncertainty.

The evolution of investment and the capital stock, in rational expectations equilibrium, depends on current expectations about the whole future path of disturbances to demand, and parameters such as the discount rate, total factor productivity, the adjustment cost function, the depreciation rate $\delta$, the size of the market and the price responsiveness of industry demand. The speed of adjustment of the capital stock depends on the discount rate and technological parameters.

The model can be generalized in a number of directions by introducing additional shocks, such as shocks to total factor productivity, labor in the production function, business taxes, externalities from capital accumulation and imperfect competition.
References

Figure 9.1
Installation Cost of Investment

![Diagram showing the relationship between installation cost and gross investment. The curve is a parabola opening upwards, indicating that the installation cost decreases as the absolute value of gross investment increases.](image-url)
Figure 9.2
The Determination of $q$ and the Equilibrium Capital Stock

$\dot{K}=0$

$\dot{q}=0$

$K_E$

$q_E$

$1$
Figure 9.3
Effects of a Permanent Rise in the Real Interest Rate
Figure 9.4
The Effects of a Rise in Total Factor Productivity