Chapter 8

Consumption and Portfolio Choice under Uncertainty

In this chapter we examine dynamic models of consumer choice under uncertainty. We continue, as in the Ramsey model, to take the decision of the household with regard to labor supply as given, assuming that each household provides a unit of labor per period. We also assume that the household can borrow and lend freely in competitive capital markets.

In intertemporal models with free borrowing and lending, consumption does not depend on current household income but on total household wealth, which consists of its current portfolio of assets, plus the present value of expected future labor income. In this sense, consumption smooths temporary changes in income, as it depends on the “permanent income” (Friedman 1957) or the “life cycle income” of the household (Modigliani Brumberg 1954).

Analyzing the consumption function of a representative household under certainty in Chapter 2, we concluded that the household consumes a proportion of its total wealth that depends on the evolution of interest rates, the pure rate of time preference, the elasticity of intertemporal substitution in consumption and the population growth rate.

Under certainty, the impact of interest rates on the propensity to consume out of total wealth depends on the elasticity of intertemporal substitution. An increase in interest rates has two kinds of effects on the propensity to consume out of total wealth. First, it induces the household to substitute future for current consumption, as it increases the cost of current consumption relative to future consumption. This is the result of intertemporal substitution in consumption. Secondly, an increase in interest rates...
increases household income from capital, inducing it to increase both current and future consumption. This is the income effect. If the intertemporal elasticity of substitution is greater than one, the propensity to consume out of total wealth decreases when interest rates rise, because the substitution effect prevails on the income effect. If the intertemporal elasticity of substitution is less than unity, the propensity to consume out of total wealth increases when interest rates rise, because the income effect prevails upon the substitution effect. Finally, in the case in which the intertemporal elasticity of substitution of consumption is equal to one, which corresponds to logarithmic preferences, the two results cancel each other out, and the propensity to consume out of total wealth is independent of the path of real interest rates, as it is equal to difference between the pure rate of time preference and the population growth rate.

In addition, an increase in real interest rates leads to a decrease in the present value of future labor income, reducing the overall wealth of the household and leading to lower consumption, even if the case where the elasticity of intertemporal substitution is equal to one. Essentially, the wealth effect of real interest rates on the present value of income from employment reinforces the substitution effect on current consumption.

The choice of consumption under conditions of uncertainty is linked to the portfolio allocation decisions of the household (Samuelson [1969], Merton [1969]).

Under uncertainty, consumption generally depends on the same factors as under certainty, only in the case of quadratic preferences, which guarantee certainty equivalence. In the case of quadratic preferences we can derive the permanent income model of consumption.

In all other cases, under uncertainty, we cannot go beyond the first order conditions and solve explicitly for consumption, unless we make further restrictive assumptions about the preferences of households or the variability of labor income.

With regard to portfolio choice, this model results in the consumption capital asset pricing model. This suggests that, under quadratic preferences, the expected return premium of a risky asset is proportional to the covariance of its return with consumption. This factor of proportionality is sometimes referred to as a consumption beta, and can be used to explain the valuation of risky assets.

It is worth keeping in mind that, under uncertainty, the permanent income hypothesis and the consumption capital asset pricing model rely on very restrictive assumptions. In addition, empirical studies suggest two puzzles that throw doubt on the validity of this model. One is the “excess sen-
sitivity” of consumption to changes in current income, and the second is the “equity premium” puzzle.

Thus, although the intertemporal model of consumption under uncertainty is a useful framework for modeling consumption and portfolio allocation decisions, one may need to go beyond the permanent income hypothesis and the consumption capital asset pricing model and consider less restrictive set ups, with precautionary savings, incomplete markets for securities and borrowing constraints.

8.1 Consumption and Portfolio Choice under Uncertainty

The problem of how consumers choose between consumption and saving under uncertainty, in conjunction with the allocation of their portfolio among different assets, was first analyzed in an intertemporal setting by Samuelson [1969] and Merton [1969]. In what follows we shall present the approach of Samuelson, who analyzed the problem in discrete time.\footnote{To understand stochastic models and decision making under uncertainty one must be aware of the concepts of random variables and stochastic process, as well as the concept of rational expectations. Appendix E introduces random variables and stochastic processes in discrete time, while appendix F discusses the concept of rational expectations and solution methods in linear stochastic models with rational expectations.}

Assume a household which at time 0 maximizes an intertemporal utility function of the form,

$$E_0 \left( \sum_{t=0}^{T-1} \left( \frac{1}{1 + \rho} \right)^t u(C_t) \right)$$

where $E_t$ denotes a mathematical expectation based on the set of available information available at time $t$. $\rho$ is the pure rate of time preference of the household and $u$ a one period utility function, depending on the level of current consumption $C_t$.

The household is uncertain regarding its future income from employment and the future returns of its portfolio of assets. The evolution of the value of the household portfolio of assets is described by,

$$A_{t+1} = (A_t + Y_t - C_t) \left[ (1 + r_t)\omega_t + (1 + x_t)(1 - \omega_t) \right]$$

$A_t$ is the value of the portfolio of assets of the household in the beginning of period $t$. $Y_t$ is labor income, which is assumed to be a random variable...
whose value is known in period \( t \). Gross savings of the household are defined by \( A_t + Y_t - C_t \). The household allocates its gross savings between a “safe” asset, with certain return \( r_t \), and a “risky” asset, with uncertain return \( x_t \). \( x_t \) is assumed to be a random variable whose value is known in period \( t \). The portfolio allocation decision of the household is determined by the percentage \( \omega \) of its assets that is invested in the “safe” asset. The term in brackets in (8.2) thus denotes the average rate of return of the household’s portfolio.

The household chooses a consumption and portfolio allocation plan for period 0, in the knowledge that it will be able to choose a new plan in the following period 1, on the basis on new information, a new plan in the following period 2, and so on, until the penultimate period \( T - 1 \). The easiest method of solving of dynamic problems under uncertainty is the method of stochastic dynamic programming.\(^2\)

Dynamic programming converts multi-period problems into a sequence of simpler two period selection problems. The first step is the introduction of a value function \( V_t(A_t) \), which is defined as,

\[
V_t(A_t) = \max \mathbb{E}_t \left( \sum_{s=t}^{T-1} \left( \frac{1}{1 + \rho} \right)^{s-t} u(C_s) \right) \tag{8.3}
\]

under the constraint (8.2).

The value function in period \( t \) is the discounted present value of the expected utility of the household, calculated under the assumption that the household follows the optimal program of consumption and portfolio allocation. This optimal value depends on the value of the portfolio of the household at the beginning of period \( t \), which is the only state variable affecting the household. The value function depends of course on the conditional joint probability distribution of the random variables that describe the future labor income of the household and the uncertain rate of return of the “risky” asset”, as well as the rate of return of the safe asset and the length of time between \( t \) and \( T - 1 \). This dependence is indicated by the time index for the value function, indicating that the value function may be changing over time.

From (8.3), the value function satisfies the following recursive equation, which is known as the Bellman equation.

\[
V_t(A_t) = \max_{\{C_t, \omega_t\}} \left\{ u(C_t) + \frac{1}{1 + \rho} E_t [V_{t+1}(A_{t+1})] \right\} \tag{8.4}
\]

\(^2\)See Appendix D for methods of intertemporal optimization.
The value function in period \( t \) is equal to the maximum utility of consumption in period \( t \) plus the discounted expected value function in period \( t + 1 \).

The first order conditions for the maximization of the right hand side of (8.4) under the constraint (8.2) are,

\[
u'(C_t) = E_t \left[ \frac{1}{1 + \rho} \left( (1 + r_t) \omega_t + (1 + x_t)(1 - \omega_t) \right) V'_{t+1}(A_{t+1}) \right]
\]

(8.5)

\[
E_t \left[ V'_{t+1}(A_{t+1})(r_t - x_t) \right] = 0
\]

(8.6)

In (8.6) we have made use of the fact that the discounted gross savings of the household, which are equal to \( (A_t - Y_t - C_t)/(1 + \rho) \), are known in period \( t \).

Applying the envelope theorem to (8.4), i.e the effects of a small change in the value of the portfolio of assets \( A_t \) on both sides of (8.4), we get that,

\[
V'(A_t) = E_t \left[ \frac{1}{1 + \rho} \left( (1 + r_t) \omega_t + (1 + x_t)(1 - \omega_t) \right) V'_{t+1}(A_{t+1}) \right]
\]

(8.7)

From (8.5) and (8.7) it follows that,

\[
V'(A_t) = u'(C_t)
\]

(8.8)

The marginal value of the household portfolio of assets in the application of the optimal program is equal to the marginal utility of consumption. As a result, we can use (8.8) to substitute the marginal utility of future consumption for the marginal value of the future portfolio of assets in the first order conditions (8.5) and (8.6).

\[
u'(C_t) = E_t \left[ \frac{1}{1 + \rho} \left( (1 + r_t) \omega_t + (1 + x_t)(1 - \omega_t) \right) u'_{t+1}(C_{t+1}) \right]
\]

(8.9)

\[
E_t \left[ u'(C_{t+1})(1 + r_t) \right] = E_t \left[ u'(C_{t+1})(1 + x_t) \right]
\]

(8.10)

Substituting (8.10) in (8.9), the two conditions take the form,

\[
u'(C_t) = \frac{1 + r_t}{1 + \rho} E_t \left[ u'(C_{t+1}) \right]
\]

(8.11)
\[ u'(C_t) = \frac{1}{1 + \rho} E_t [(1 + x_t) u'(C_{t+1})] \quad (8.12) \]

Conditions (8.11) and (8.12) have a simple interpretation, which is a generalization of the interpretation of the Euler equation for consumption in the Ramsey problem. Recall, that the Euler equation for consumption in the Ramsey problem suggests that the marginal rate of substitution between the levels of consumption in the two periods must be equal to the marginal rate of transformation.

In the case of (8.11), assume that the household reduces its consumption by an infinitesimally small amount \( dC \) in period \( t \), invests the amount in the “safe” asset, and consumes the return in the next period \( t + 1 \). The reduction in its utility in period \( t \) is equal to \( u'(C_t) \), that is the left hand side of (8.11). The increase in its expected utility in period \( t + 1 \) is equal to the right hand side of (8.11). In the application of the optimal program, this infinitesimal reallocation does not affect the value of the plan, and as a result (8.11) holds. (8.12) holds for the same reason, under the assumption that the household invests in the “risky” asset, rather than the “safe” asset in its portfolio.

### 8.1.1 The Random Walk Model of Consumption

Equations (8.11) and (8.12) are just first order conditions, and do not describe the full solution of the problem. Nevertheless, they suggest strong restrictions for the dynamic behavior of consumption. (8.11) implies that,

\[ \frac{1 + r_t}{1 + \rho} u'(C_{t+1}) = u'(C_t) + \varepsilon_{t+1} \quad (8.13) \]

where \( E_t(\varepsilon_{t+1}) = 0 \).

(8.13) suggests that given the marginal utility of consumption \( u'(C_t) \), there no additional information available in period \( t \) that could help predict \( u'(C_{t+1}) \), the future marginal utility of consumption.

Assuming that the utility function is quadratic in consumption, and that the rate of return of the safe asset is equal to the pure rate of time preference, then (8.13) takes the form,

\[ C_{t+1} = C_t + \varepsilon_{t+1} \quad (8.14) \]

The model implies that consumption follows a “random walk” process. Given the level of consumption in period \( t \), no other variable known in period \( t \) can help predict consumption in period \( t + 1 \).
This prediction of the model was first highlighted by Hall [1978], who also investigated it empirically.

### 8.1.2 The Consumption Capital Asset Pricing Model

Conditions (8.11) and (8.12) can also be used to determine the rate of return of the risk free asset and the expected rate of return, and hence the price of the risky asset. This requires that all individual households are alike, i.e. that there is a representative household.

From (8.11), the rate of return of the risk free asset will satisfy,

\[
1 + r_t = (1 + \rho) \frac{u'(C_t)}{E_t [u'(C_{t+1})]} \tag{8.15}
\]

From (8.12), it follows that,

\[
u'(C_t) = \frac{1}{1 + \rho} \left\{ E_t (1 + x_t) E_t (u'(C_{t+1})) + \text{Cov}_t (1 + x_t, u'(C_{t+1})) \right\} \tag{8.16}
\]

From (8.16), the expected rate of return of the risky asset will satisfy,

\[
E_t (1 + x_t) = (1 + \rho) \frac{u'(C_t)}{E_t (u'(C_{t+1}))} - \frac{\text{Cov}_t (1 + x_t, u'(C_{t+1}))}{E_t (u'(C_{t+1}))} \tag{8.17}
\]

From (8.15) and (8.17), the expected return premium of the risky asset is given by,

\[
E_t (x_t) - r_t = - \frac{\text{Cov}_t (1 + x_t, u'(C_{t+1}))}{E_t (u'(C_{t+1}))} \tag{8.18}
\]

The expected return premium of the risky asset depends on the covariance of the rate of return of the risky asset with the marginal utility of consumption. Given that the marginal utility of consumption is negatively correlated with consumption, because of decreasing marginal utility, the expected return premium of the risky asset will depend positively on the covariance of the rate or return of the risky asset with consumption. Risky assets whose returns are positively correlated with consumption, will tend to have a higher expected return relative to the risk free asset.

This model of the determination of expected asset returns is known as the consumption capital asset pricing model, or, consumption CAPM.
8.2 Full Analysis of Consumption and Portfolio Choice

From the first order conditions one cannot fully describe the behavior of consumption and savings, apart from specific cases. There are two special cases where we can come up with specific solutions. The first is the case of insurable income risk, and the second is the case of quadratic utility functions.

As demonstrated by Merton [1971], if labor income can be insured, we can deduce specific solutions for consumption for a broad class of utility functions, the so called hyperbolic absolute risk aversion, or HARA, utility functions. This class includes isoelastic utility functions with constant relative risk aversion (constant relative risk aversion, or CRRA), the exponential utility function with constant absolute risk aversion (constant absolute risk aversion or CARA) and quadratic utility functions.

One way to derive the specific solution is to use the Belmann principle of optimality, which says that for any value of the state variable (the portfolio of assets in this case) at a given time period, the solution for the future must be optimal. Using this principle and the value function, the solution can be found through backward induction. For example, in period \( T - 2 \), for any value of the portfolio of assets \( A_{T-2} \), the household faces a two-period problem. Solving this problem, take a step back, and solve the problem of the period \( T - 3 \), having already identified the value of the value function of \( T - 2 \). Then we move on and solve the same problem inductively for the \( T - 4 \) period and so on.

In the case of an infinite time horizon we take the limit of the solution to the problem of \( T \) periods , as \( T \) tends to infinity. Alternatively, we can also solve the problem of infinite periods directly.

For example, if the consumer has an infinite time horizon, if the “safe” interest rate is fixed and if the “uncertain” return \( x_t \) is distributed according to an independent, uniform, probability distribution, the value function is independent of time and only depends on the state variable \( A_t \). Therefore, we can presume its form, derive the consumption function and verify if our presumption was correct.

The reason that HARA type utility functions allow us to infer analytical solutions, is that the value function belongs to the same family as the utility function, and all that remains is to infer the parameters of the value function.
8.2.1 The Case of Logarithmic Preferences

Consider the simple case in which,
\[ u(C_t) = \ln C_t \]  
\[ (8.19) \]

Using the result of Merton that the value function has the same functional form as the utility function, for HARA type utility functions, we presume that the value function takes the form,
\[ V(A_t) = a \ln(A_t) + b \]  
\[ (8.20) \]

where \( a \) and \( b \) are constant parameters to be determined. This conjecture allows us to formulate the maximization problem in period \( t \) as,
\[ \max \ln(C_t) + \frac{1}{1+\rho} E_t [a \ln(A_{t+1}) + b] \]  
\[ (8.21) \]

under the constraint,\[ A_{t+1} = (A_t - C_t) [(1 + r)\omega_t + (1 + x_t)(1 - \omega_t)] \]  
\[ (8.22) \]

Solving for consumption \( C \) and the share of the portfolio invested in the “safe” asset \( \omega \) we get,
\[ C_t = \left(1 + \frac{a}{1+\rho}\right)^{-1} A_t \]  
\[ (8.23) \]

\[ E_t \left[(r - x_t)((1 + r)\omega + (1 + x_t)(1 - \omega))^{-1}\right] = 0 \]  
\[ (8.24) \]

(8.23) determines consumption as a linear function of the value of the total portfolio of assets of the household. (8.24) determines indirectly the optimal proportion invested in the “safe” asset \( \omega \) as a constant, due to the assumption that the return of the “risky” asset \( x \) is distributed according to a uniform, independent probability distribution. The proportion \( \omega \) is independent of the total value of the portfolio of assets.

In order to find \( a \) and \( b \) we substitute (8.22), (8.23) and (8.24) in the value function (8.21) and compare coefficients with (8.20). \( b \) is a complex but not economically significant constant, which depends on all the parameters of the model. Using (8.8), \( a \) is determined as \((1 + \rho)/\rho\). Substituting this value in the consumption function (8.23), we get,

\[ ^3 \text{Since we assume that labor income risk is insurable, we concentrate on the case in which labor income } Y \text{ is zero.} \]
\[ C_t = \frac{\rho}{1 + \rho} A_t \]  

(8.25)

In conclusion, assuming that labor income is insurable and that the utility function is logarithmic, we get a consumption function analogous to the case of certainty. Consumption is a linear function of the total value of the portfolio of the household, and the marginal propensity to consume out of wealth depends only on the pure rate of time preference and not on the real interest rate or the rate of return of the “risky” asset. Changes in future non labor income (dividends and interest) affect consumption only through their impact on total wealth.

8.2.2 Quadratic Preferences and Certainty Equivalence

The second case we shall examine is the case of quadratic preferences. We shall initially assume that the portfolio consists only of the “safe” asset. The maximization problem of the household is,

\[ \text{max } E_0 \left[ \sum_{t=0}^{T-1} \left( \frac{1}{1 + \rho} \right)^t \left( aC_t - bC_t^2 \right) \right] \]  

(8.26)

under the constraints,

\[ A_{t+1} = (1 + r) \left( A_t + Y_t - C_t \right), A_T \geq 0 \]  

(8.27)

From the first order conditions for a maximum,

\[ E_t C_{t+1} = \frac{r - \rho}{1 + r} \frac{a}{2b} + \frac{1 + r}{1 + \rho} C_t \]  

(8.28)

In what follows we shall assume that \( r = \rho \). In this case, (8.28) takes the form,

\[ E_t C_{t+1} = C_t \]  

(8.29)

(8.29) implies that,

\[ E_0 C_t = C_0, t = 1, 2, ..., T - 1 \]  

(8.30)

Because of the equality between the real interest rate and the pure rate of time preference, the optimal consumption path is such that there is \textit{perfect consumption smoothing}. The optimal path of consumption is such that expected consumption is constant along the optimal path.
The budget constraint (8.27) implies that,

\[ A_T = A_0 (1 + r)^T + \sum_{t=0}^{T-1} (1 + r)^{T-t}(Y_t - C_t) \]  

(8.31)

Because the household does not derive utility from its portfolio directly, but only from its consumption, its consumption in the last period will be such that \( A_T = 0 \). Using this transversality condition in (8.31), the intertemporal budget constraint of the household will be equal to,

\[ \sum_{t=0}^{T-1} \left( \frac{1}{1 + r} \right)^t C_t = A_0 + \sum_{t=0}^{T-1} \left( \frac{1}{1 + r} \right)^t Y_t \]  

(8.32)

According to (8.32), the present value of consumption of the household is equal to the value of its initial portfolio, plus the present value of expected labor income. The household knows at time 0 that the budget constraint (8.32) should be satisfied, but does not know future labor income. Accordingly, at time 0 the household aims to satisfy,

\[ E_0 \left[ \sum_{t=0}^{T-1} \left( \frac{1}{1 + r} \right)^t C_t \right] = A_0 + E_0 \left[ \sum_{t=0}^{T-1} \left( \frac{1}{1 + r} \right)^t Y_t \right] \]  

(8.33)

(8.33) suggests that the expected present value of consumption is equal to the value of the original portfolio of household, plus the present value of expected labor income.

Substituting (8.30) in (8.33), and taking the limit as \( T \) tends to infinity,

\[ C_0 = \frac{r}{1 + r} \left( A_0 + E_0 \left[ \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t Y_t \right] \right) \]  

(8.34)

Consumption is constant and is a fixed percentage of the total wealth of the household, including the present value of expected labor income. In every period, the household consumes a constant fraction of its total wealth, depending on the real interest rate (or the pure rate of time preference), so that expected total wealth remains constant.

### 8.2.3 The “Permanent Income” Hypothesis with Quadratic Preferences

From (8.30), the change in consumption from period to period is determined only by the revision of expectations regarding labor income.
\[ C_t - C_{t-1} = \frac{r}{1 + r} \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^i (E_t (Y_{t+i}) - E_{t-1} (Y_{t+i})) \] (8.35)

For example, if labor income follows a stationary first order autoregressive stochastic process, i.e. an AR(1) process of the form,

\[ Y_t = Y_0 + \lambda Y_{t-1} + \varepsilon_t \] (8.36)

where \( 0 < \lambda < 1 \) and \( \varepsilon \) a “white noise” process, then, from (8.31), the change in current consumption depends only on the current innovation in labor income \( \varepsilon \). Thus, under the assumption in (8.36), (8.35) takes the form,

\[ C_t - C_{t-1} = \frac{r}{1 + r - \lambda} \varepsilon_t \] (8.37)

The coefficient of the transitory innovation in income \( \varepsilon \) is smaller than unity. (8.37) incorporates the predictions of the “permanent income” hypothesis of Friedman [1957] and the “life cycle” hypothesis of Modigliani and Brumberg [1954], that consumption smooths out transitory changes in income.

If \( \lambda \) is equal to unity, then disturbances in labor income \( \varepsilon \) are of a permanent nature, and the coefficient of (8.33) is also equal to unity. Permanent changes in labor income lead to equivalent permanent changes in household consumption.

Empirical studies of the “permanent income” hypothesis suggest that aggregate consumption displays excess sensitivity to changes in current income, compared to the predictions of this hypothesis. This evidence suggests that one may have to go beyond the “permanent income” hypothesis in order to adequately account for aggregate consumption.4

8.2.4 The Consumption Capital Asset Pricing Model with Quadratic Preferences

Assuming quadratic preferences, as we have in (8.26), the marginal utility of consumption is given by,

\[ \text{4The recent empirical literature on the “permanent income” hypothesis is huge. Tests based on aggregate data include the original paper of Hall [1978], Flavin [1981], Flavin [1985], Hansen and Singleton [1982], Hansen and Singleton [1983], Campbell and Mankiw [1989], Campbell and Mankiw [1991] and others. Attanasio [1999] surveys both aggregate and disaggregated studies.} \]
The marginal utility of consumption is thus a negative linear function of consumption.

Substituting for the marginal utility of consumption in (8.18), we get,

$$E_t (x_t) - r_t = \frac{2b\text{Cov}_t (1 + x_t, C_{t+1})}{a - bE_t C_{t+1}}$$

(8.39)

This confirms that under quadratic preferences the expected return premium of a risky asset is proportional to the covariance of its return with consumption. This factor of proportionality is sometimes referred to as a consumption beta, from a regression of consumption growth on asset returns. Thus, a central prediction of the consumption CAPM is that the return premium of a risky asset is proportional to its consumption beta.  

However, empirical studies suggest a so called equity premium puzzle, i.e. a much bigger difference between the average return of equities (the risky asset) and government bonds (the safe asset) than would be suggested by the consumption CAPM.  

### 8.3 Precautionary Savings and Borrowing Constraints

The assumption of quadratic preferences is quite restrictive. When preferences are not quadratic, certainty equivalence does not hold, and uncertainty about non-insurable labor income generally affects consumption.

To see this, consider for example the first order condition (8.11) with a constant risk free rate $r$.

$$u'(C_t) = \frac{1 + r}{1 + \rho} E_t [u'(C_{t+1})]$$

(8.40)

So long as consumers are risk averse, i.e. so long as $u'' < 0$, increased uncertainty in the form of an increase in the variance of consumption, decreases expected utility. But the effect of increased uncertainty on behavior

\footnote{See Merton [1973] and Breeden [1979]. The original capital asset pricing model (CAPM) of Sharpe [1964] and Lintner [1965] assumed that investors are concerned with the mean and variance of the return of their portfolio, rather than the mean and variance of consumption. That version of the model thus focused on so-called market betas, that is coefficients from regressions of asset returns on the returns of a market portfolio.}

\footnote{See the important paper of Mehra and Prescott [1985] that has generated a large theoretical and empirical literature on this issue.}
depends on whether it affects the expected marginal utility of consumers, through the first order condition (8.40).

As long as utility is quadratic, marginal utility is linear in consumption, and \( u'' = 0 \). Thus, in the case of quadratic utility, the variance of consumption has no effect on expected marginal utility, and thus no effect on optimal behavior. After all, this is why certainty equivalence holds with quadratic preferences. In the general case however, for most plausible utility functions, \( u''' > 0 \). This means that marginal utility is convex in consumption, and an increase in uncertainty increases expected marginal utility. Thus, from (8.40), increased variability of future consumption would require an increase in expected future consumption relative to current consumption. Uncertainty leads consumers to defer consumption, and thus be more prudent. The effects of prudence on savings were first analyzed by Leland [1968], and subsequently by Sandmo [1970] and Dreze and Modigliani [1972].

However, in general it is almost impossible to solve for optimal consumption in the presence of prudent behavior. A case that can be solved analytically is the case of constant absolute risk aversion, which implies a utility function of the form,

\[
u(C_t) = \frac{1}{\theta} e^{-\theta C_t} \tag{8.41} \]

where \( \theta \) is the coefficient of absolute risk aversion. This case has been analyzed in Caballero [1990].

Non quadratic preferences and precautionary savings are not the only deviations from the permanent income hypothesis that have been considered in the literature. Other deviations that have been considered are incomplete markets and liquidity (borrowing) constraints, as well as departures from full optimization. These extensions have been motivated by empirical weaknesses of the permanent income hypothesis and the consumption capital assets pricing model, and are surveyed extensively in Attanasio [1999]. Yet, because of the complexity of such models, they have not been integrated in macroeconomic models.

### 8.4 Conclusions

In this chapter we have examined the determination of household consumption under conditions of uncertainty, in conjunction with the determination of the allocation of the portfolio of the household among alternative assets (Samuelson [1969], Merton [1969]).
Under conditions of uncertainty, for a household that can borrow and lend freely in the capital market, consumption generally depends on the same factors as under certainty. The current and expected future rates of assets, the current and expected future labor income and the total value of the portfolio and human wealth of the household.

Consumption does not depend on current household income but on total wealth, which consists of the value of its portfolio, plus the present value of current and expected future labor income. In this sense, consumption smooths out temporary changes in income, as it depends on “permanent” or “life cycle” income. (Friedman [1957], Modigliani and Brumberg [1954]).

However, the “permanent” income hypothesis cannot explain many of the features of individual or aggregate consumption patterns, especially the “excess sensitivity” of consumption to changes in current income. In addition, the consumption capital asset pricing model, which is an associated prediction of stochastic intertemporal models of consumption, seems to be refuted by the “equity premium” puzzle. The literature has thus examined models that result in precautionary savings, or in which markets are incomplete and households are also bound by borrowing constraints (see Attanasio [1999]). Such models are quite complex and hard to integrate with dynamic general equilibrium macroeconomic models.