Chapter 2
Savings, Investment and Economic Growth

The analysis of why some countries have achieved a high and rising standard of living, while others have been left behind, is one of the major challenges of economics in general, and macroeconomics in particular.

In this chapter we begin to investigate the determinants of long-run economic growth. We focus primarily on the relationship between savings, investment, physical capital accumulation and economic growth.

The starting point for the analysis of this process is the Solow (1956) model. This model is based on a neoclassical production function and the assumption of a constant savings rate. Given that in a closed economy savings are equal to investment, the process of capital accumulation depends on the savings rate which determines the investment rate.\(^1\)

In this model, capital accumulation per worker continues until savings per employee are equated with depreciation and the additional investment required to maintain a constant ratio of capital to labor.

In the case where technological progress raises labor productivity continuously, then capital accumulation per efficiency unit of labor continues until savings per efficiency unit of labor are equated to depreciation plus the additional investment required to provide for population growth and the rate of technical progress.

In the long-run equilibrium of this model, alternatively referred to as the steady state or the balanced growth path, economic growth is exogenous and equal to the rate of population growth plus the rate of technical progress, which raises the efficiency of labor. Essentially, in the long-run equilibrium, per capita output increases at the exogenous rate of technical progress.

During the adjustment process to the balanced growth path, an economy that has a low initial capital stock, has a growth rate which is higher than the long-run growth rate. Capital accumulates at a rate that exceeds the sum of the rate of growth of population and the rate of technical progress. For an economy that has a high initial capital stock, the growth rate is below the long-run growth rate, as capital accumulates at a rate that falls short of the sum of the rate of growth of population and the rate of technical progress.

This model predicts that economies converge to a balanced growth path. A "poor" economy, in terms of its initial capital stock, and a “rich” economy, again in terms of its initial capital stock, converge

\(^1\) This model is frequently referred to as the Solow-Swan model, as a similar model was introduced at in the same year by Swan (1956).
to the same balanced growth path, provided that they are characterized by the same savings rate and the same technological and demographic parameters.

However, if two economies have different savings rates, different total factor productivity, different initial labor efficiency, different rates of population growth or a different depreciation rate of capital, they will converge to different balanced growth paths. Convergence in this model is conditional, and the conditions are related to the structural characteristics of different economies, such as their savings and investment rates, total factor productivity, the rate of population growth and the rate of technical progress.

This model predicts that a higher savings (and investment) rate results in higher steady state capital and output per worker. Furthermore, it predicts a positive impact on capital and output per worker from higher total factor productivity and initial labor efficiency, and a negative impact from the rate of population growth, the rate of technical progress and the depreciation rate of capital.

The Solow model is a key model and an important reference point in the theory of economic growth. Although its roots lie in older models, and although it has a number of theoretical and empirical weaknesses, this model provides a very useful, simple and flexible framework for the analysis of the growth process.

However, accumulation of physical capital, which is the main engine of economic growth in the Solow model, cannot fully explain either the long-term growth of output per worker and per capita income that has been observed in developed economies, or the large differences in labor productivity and living standards per head between developed and less developed economies.

Only a small part of these differences can be explained by the accumulation of physical capital. Most of it is accounted for by differences in total factor productivity and technical progress, which in the Solow model are considered exogenous parameters. In this sense, the Solow model, like all models that rely on similar assumptions about technology and technical progress, shows us how to overcome its weaknesses and to try to explain technical progress endogenously.

2.1 The Solow Model

In order to account for the process of economic growth, the Solow model focuses on three main endogenous variables. Total output \(Y\), the total physical stock \(K\) and aggregate consumption \(C\). Two additional endogenous variables, the real wage \(w\) and the real interest rate \(r\), are determined if one assumes competitive markets for factors of production. The number of employees \(L\) is assumed equal to an exogenously evolving total population, and the efficiency of labor \(h\) is assumed to evolve exogenously as well.

Thus, the rate of growth in the number of employees is equal to the population growth rate \(n\) and is considered exogenous. The rate of growth in the efficiency of labor is equal to the rate of exogenous technical progress \(g\).

The model explains the level and rate of growth of output and physical capital as functions of these exogenous factors \((n\) and \(g\)), the saving rate \((s)\), which is also considered exogenous, total factor productivity and the exogenous rate of depreciation of capital \((\delta)\). After the capital stock, output, consumption and investment are determined, one can determine the real interest rate \(r\).
(renumeration of capital) and real wages \( w \) (remuneration of labor), as these depend on the ratio of capital to total labor efficiency.

### 2.1.1 The Neoclassical Production Function

At each point in time \( t \), the economy has a stock of capital, number of employees and labor efficiency, which are combined to produce output. The production function takes the form,

\[
Y(t) = F(K(t), h(t) L(t))
\]  

(2.1)

It is worth noting the following characteristics of the neoclassical production function:

First, time \( t \) enters the production function solely through the factors of production \( K(t) \) and \( h(t) L(t) \). Output can change over time only through changes in the factors of production.

Second, technical progress is assumed to increase only the efficiency of labor. This is called labor augmenting technical progress, or Harrod neutral technical progress.

Third, the production function is characterized by constant returns to scale. Multiplying all factors of production by any number, multiplies the scale of production by the same number.

Because of the assumption of constant returns to scale, the production function can be written as,

\[
y(t) = f(k(t))
\]  

(2.2)

where,

- \( y = Y/hL \) Output per efficiency unit of labor
- \( k = K/hL \) Capital per efficiency unit of labor
- \( f(k) = F(k, 1) \) Production function per efficiency unit of labor

(2.2) is often referred to as the production function in intensive form. The intensity of production (output per efficiency unit of labor) depends on capital intensity (capital per efficiency unit of labor).

Fourth, it is assumed that the production function satisfies the following properties:

\[
f(0) = 0, f’ = \frac{\partial f}{\partial k} > 0, f” = \frac{\partial^2 f}{\partial k^2} < 0
\]

The marginal product of capital intensity is positive but declining. The production function in intensive form, with these additional assumption is depicted in Figure 2.2.

Finally, it is assumed that,

\[
\lim_{k \to 0} f’(k) = \infty, \lim_{k \to \infty} f’(k) = 0
\]
These final assumptions are called the Inada (1964) conditions, and are stronger than what is required for the central predictions of the Solow model. The Inada conditions ensure that the marginal product of capital is very high when capital intensity is small, and very small when capital intensity is high, and are required in order to prove the global uniqueness of the balanced growth path.

2.1.2 The Cobb Douglas Production Function

A particular production function which is often used in the theory of growth, but also more generally in macroeconomics, is the Cobb Douglas production function. This takes the form,

\[ F(K(t), h(t)L(t)) = AK(t)^\alpha (h(t)L(t))^{1-\alpha}, \quad A > 0, \quad 0 < \alpha < 1 \]  

(2.3)

where \(A\) is defined as total factor productivity, and \(\alpha\) as the exponent (share) of capital in total production. \(1-\alpha\) is the corresponding exponent (share) of labor.

The Cobb Douglas production function in intensive form is given by,

\[ y(t) = f(k(t)) = Ak(t)^\alpha \]  

(2.4)

One can easily confirm that the Cobb Douglas production function (2.3) satisfies all the assumptions we have made about the neoclassical production function. The marginal product of capital and labor are positive and declining, and the Inada conditions are satisfied.

In addition, for the Cobb Douglas production function, labor augmenting technical progress (Harrod neutral) does not differ from capital augmenting technical progress, or technical progress that augments both factors (Hicks neutral). The reason is that in the Cobb Douglas production function the factors of production enter multiplicatively, and thus, it does not matter which of the factors of production is multiplied by technical progress.

2.1.3 Population Growth and Technical Progress

We shall analyze the Solow model in continuous time, assuming that \(i\) is a continuous variable.\(^2\)

At time 0, the initial levels of capital, number of employees and efficiency of labor are given.

We shall assume that the number of employees is a constant fraction of total population, and grows continuously at the (exogenous) rate of population growth \(n\).

\[ L(t) = L(0)e^{nt} \]  

(2.5)

where \(L(0)\) is the number of employees at time 0, and \(e\) is the base of natural logarithms.

We shall also assume that the efficiency of labor also grows continuously at the exogenous rate of technical progress \(g\).

\(^2\) In the Annex to this Chapter we also analyze the Solow model in discrete time, where \(t=0,1,2,...\) is an integer, that refers to discrete time periods, like years, months, weeks, days or hours.
where $h(0)$ is the efficiency of labor at time 0.

From (2.5) and (2.6) it follows that,

\[
\dot{L}(t) = \frac{dL(t)}{dt} = nL(t) \quad (2.7)
\]

\[
\dot{h}(t) = \frac{dh(t)}{dt} = gh(t) \quad (2.8)
\]

A dot on top of a variable denotes its first derivative with respect to time, i.e. its change over time.\(^3\)

### 2.1.4 Savings, Capital Accumulation and Economic Growth

The output produced is income to households, which is either consumed or saved. In the Solow model, the share of income which is saved is assumed exogenous, and denoted by $s$.

Consumption $C$, is thus given by,

\[
C(t) = (1 - s)Y(t), \quad 0 < s < 1 \quad (2.9)
\]

The demand for total output consists of consumption plus gross investment $I$.

\[
Y(t) = C(t) + I(t) \quad (2.10)
\]

(2.10) is an equilibrium condition in the product market, stating that total production (output supply) is equal to the demand for output.

Gross investment consists of additions to the capital stock, plus replacement investment, and is given by,

\[
I(t) = \dot{K}(t) + \delta K(t), \quad 0 \leq \delta \leq 1 \quad (2.11)
\]

where $\delta$ is the exogenous rate of depreciation of the capital stock.

Substituting the consumption function (2.9) and the definition of gross investment (2.11), in the equilibrium condition (2.10), we get,

\[
Y(t) = (1 - s)Y(t) + \dot{K}(t) + \delta K(t) \quad (2.12)
\]

Solving (2.12) for the change in the capital stock, we get,

\[^3\text{Technically, (2.7) and (2.8) are first order linear differential equations, whose solution is given by (2.5) and (2.6) respectively. For an introduction to differential equations see Mathematical Annex 2.}\]
From (2.13), the accumulation of capital is determined by the difference between savings and replacement investment. To the extent that savings is higher than replacement investment, the capital stock grows over time. If savings is lower than replacement investment, the capital stock is reduced over time.

Dividing (2.13) through by \(hL\), taking into account that \(L\) grows at a rate \(n\), and \(h\) grows at a rate \(g\), we get,

\[
\dot{k}(t) = sy(t) - (n + g + \delta)k(t)
\]

(2.14) is the capital accumulation equation expressed in efficiency units of labor. To the extent that savings per efficiency unit of labor is higher than the investment required to keep capital per efficiency unit of labor constant, capital per efficiency unit of labor grows over time. In the opposite case, it falls over time.

Using the production function in intensive form, (2.2), to replace for \(y\) in (2.14), we get,

\[
\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t)
\]

The non linear differential equation (2.15) is the key equation of the Solow model. It suggests that the change over time in capital per efficiency unit of labor is determined by the difference in two terms that both depend on the level of capital per efficiency unit of labor. The first term is current savings and investment per efficiency unit of labor, and the second term is equilibrium investment per efficiency unit of labor. Equilibrium investment is defined as the investment rate that keeps capital per efficiency unit of labor constant.

2.1.5 The Balanced Growth Path and the Convergence Process

Given that the total efficiency of labor is increasing at an exogenous rate \(n+g\), and that a fraction \(\delta\) of the capital stock needs to be replaced at every moment, due to depreciation, the investment that is required to keep the capital stock per efficiency unit of labor constant is given by \((n+g+\delta)k\). This we shall denote as equilibrium investment.

Equilibrium capital per efficiency unit of labor is thus determined by,

\[
\dot{k}(t) = 0, \Rightarrow sf(k(t)) = (n + g + \delta)k(t)
\]

We shall refer to this equilibrium level of capital intensity as steady state capital per effective unit of labor \(k^*\), and one can easily deduce that \(k^*\) is constant and independent of time. \(k^*\) defines the balanced growth path or steady state, as all other steady state variables in the model depend on \(k^*\).

On the balanced growth path, the steady state capital stock, output, consumption and investment per efficiency unit of labor are constant. The per capita steady state capital stock, output, consumption
and investment are all growing at the exogenous rate of technical progress $g$. The aggregate capital stock, output, consumption and investment are growing at the rate $g+n$, which is the sum of population growth and the rate of technical progress.

The determination of $k^*$, and the dynamic adjustment of $k$ towards $k^*$ are depicted in the phase diagram in Figure 2.2. The straight line depicts equilibrium investment $(n+g+\delta)k$. The curved line $sf(k)$ depicts current savings and investment. At the point $k^*$, current savings and investment are equal to equilibrium savings and investment.

To the left of $k^*$ current investment is higher than equilibrium investment, and $k$ is increasing over time. To the right of $k^*$ current investment is lower than equilibrium investment, and $k$ is declining over time.

The equilibrium at $k^*$ is unique and globally stable. Irrespective of initial conditions, the economy converges to $k^*$, which is the equilibrium capital stock per efficiency unit of labor.

In conclusion, the Solow growth model does not explain long-run economic growth, i.e economic growth along the balanced growth path, as this is equal to the sum of two exogenous parameters, $g$ and $n$. It does not explain the growth of per capita income and consumption along the balanced growth path either, as these are equal to the rate of exogenous technical progress $g$.

What the Solow model does explain is the level of the per capita capital stock and per capita output and income, the level of per capita consumption and real wages and the real interest rate, on the balanced growth path. These depend on all the parameters of the model, as we shall shortly see.

In addition, the Solow growth model explains the process of convergence towards the balanced growth path. The process of convergence predicted by the model is the result of the accumulation of physical capital. The growth rate of output, or output per capita, in the convergence process differs from the long run growth rate $g+n$ or $g$, to the extent that, during the convergence process, the economy accumulates capital at a different rate than $n+g$.

2.2 The Significance of the Savings Rate and the Golden Rule

In the Solow model one can prove that a rise in the savings rate results in an increase in steady state capital and output. It can also be shown that the rate of growth of the per capita capital stock and per capita output and income increase temporarily above the rate of long-run economic growth $g+n$. The relevant analysis is presented in Figure 2.3.

2.2.1 The Savings Rate and the Balanced Growth Path

We assume that the initial balanced growth path is at $(y^*, k^*)$ in Figure 2.3. A rise in the savings rate from $s$ to $s'$ leads to an increase in savings and investment that initiates a process of capital accumulation, which gradually causes an increase in output and income per effective unit of labor. The economy starts converging to a new balanced growth path $(y^{**}, k^{**})$ which is characterized by both higher capital and higher income. During the adjustment process, savings and investment exceed equilibrium investment, and the rate of growth exceeds the long-run growth rate $g+n$.
2.2.2 The Savings Rate and the Golden Rule

Capital and income increase unequivocally following a rise in the savings rate. What happens to consumption is more uncertain, as a rise in the savings rate reduces consumption for any given level of income. Initially, as capital, output and income are given, a rise in the savings rate causes a temporary fall in consumption. Gradually capital output and income rise and so does consumption. Whether in the new balanced growth path steady state consumption per capita will be higher or lower than in the original balanced growth path depends on the difference between the marginal product of capital and $n + g + \delta$, the latter being the marginal increase in equilibrium investment. Consumption will be higher in the new balanced growth path if the marginal product of capital is higher than $n + g + \delta$, and it will be lower in the opposite case.

To see this, recall that steady state consumption is given by,

$$c^* = f(k^*) - (n + g + \delta)k^*$$  \hspace{1cm} (2.17)

It follows that the change in steady state consumption following a rise in the savings rate is given by,

$$\frac{\partial c^*}{\partial s} = (f'(k^*) - (n + g + \delta))\frac{\partial k^*}{\partial s}$$  \hspace{1cm} (2.18)

Since the last term in the right hand side of (2.18) has been shown to be positive, the impact of the change in the savings rate on steady state consumption per effective unit of labor depends on the difference between the marginal product of capital $f'(k^*)$ from the equilibrium investment rate $n + g + \delta$.

Another way to express this is to say that the change in steady state consumption depends on the difference between the net (of depreciation) marginal product of capital $f'(k^*) - \delta$ and the long-run growth rate $g + n$.

If the net marginal product of capital is smaller than the long-run growth rate, then the extra product from the accumulation of capital will not be sufficient to fund the higher equilibrium investment rate, and consumption will have to go down. If the net marginal product of capital is higher than the long-run growth rate, then the extra product from the accumulation of capital will be more than sufficient to fund the higher equilibrium investment rate, and consumption will also increase.

In the special case where the net marginal product of capital at the original balanced growth path is exactly equal to the long-run growth rate, equilibrium consumption will remain unchanged following a rise in the savings rate.
In the latter case, equilibrium consumption is at its highest possible level, and the value of $k^*$ that corresponds to this case is referred to as the \textit{golden rule} capital stock.

The golden rule capital stock is defined as the steady state capital stock (per effective unit of labor) that maximizes steady state consumption (per effective unit of labor). Since the welfare of households is usually assumed to depend on consumption, the maximization of steady state consumption per effective unit of labor is a reasonable proxy for the maximization of steady state welfare. From (2.17), the first order conditions for the maximization of consumption require,

$$f'(k^*) = n + g + \delta \Rightarrow f'(k^*) - \delta = n + g \quad (2.19)$$

From (2.19), the steady state capital stock that maximizes steady state consumption is the one that results in a net marginal product of capital equal to the long-run growth rate. This is the golden rule capital stock.

\subsection*{2.2.3 The Elasticity of Steady State Output with Respect to the Savings Rate}

One can show that the long run elasticity of output with respect to the savings rate is equal to the ratio of the share of capital to the share of labor in total output.

To prove this, we start from the change in steady state output following a change in the savings rate. This is equal to,

$$\frac{\partial y^*}{\partial s} = f'(k^*) \frac{\partial k^*}{\partial s} \quad (2.20)$$

$k^*$ is defined by,

$$sf(k^*) = (n + g + \delta)k^* \quad (2.21)$$

Differentiating (2.21) with respect to $s$, we get,

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n + g + \delta) - sf'(k^*)} \quad (2.22)$$

Substituting (2.22) in (2.20), we get,

$$\frac{\partial y^*}{\partial s} = \frac{f'(k^*)f(k^*)}{(n + g + \delta) - sf'(k^*)} \quad (2.23)$$

From (2.23), the long-run elasticity of output with respect to the savings rate is given by,

$$\frac{s}{y^*} \frac{\partial y^*}{\partial s} = \frac{s}{f(k^*)} \frac{f'(k^*)f(k^*)}{(n + g + \delta) - sf'(k^*)} = \frac{(n + g + \delta)k^* f'(k^*)}{f(k^*)(n + g + \delta)(1 - k^* f'(k^*) / f(k^*))} \quad (2.24)$$

Using (2.21) to replace $n + g + \delta$, (2.24) can be re-written as,
where \( a_k(k^*) \) is the elasticity of total output with respect to capital, at the steady state. With competitive markets factor incomes are equal to their marginal products. In such a case, the elasticity of total output with respect to capital is equal to the share of capital in total output. A commonly accepted estimate of the share of capital in total output is 1/3. Using this estimate, the long run elasticity of total output with respect to the savings rate is equal to 1/2.

### 2.3 The Speed of Convergence towards the Balanced Growth Path

Near the balanced growth path, the speed of convergence of \( k \) towards \( k^* \) depends on their distance. On the basis of widely accepted values for the parameters of the model, one can show that the speed of convergence in the Solow model is about 4% per annum. As a result, the Solow model predicts that it should take slightly above 17 years to close half of the gap between \( k \) and \( k^* \).

In order to derive the speed of convergence we start from the basic accumulation equation of the Solow model.

\[
\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t) \tag{2.26}
\]

Steady state capital (per effective unit of labor) \( k^* \) is determined from (2.26) for \( \dot{k}(t) = 0 \).

In order to determine the speed at which \( k(t) \) approaches \( k^* \), we linearize (2.26) around \( k^* \). From the linear Taylor approximation of the non-linear differential equation (2.26) around \( k^* \), we get,

\[
\dot{k}(t) \approx \left( \frac{\partial k}{\partial k} \right)_{k=k^*} (k(t) - k^*) \tag{2.27}
\]

where the first derivative is taken from (2.26).

(2.27) can be written as,

\[
\dot{k}(t) \approx -\lambda (k(t) - k^*) \tag{2.28}
\]

where \( \lambda = -\left( \frac{\partial k}{\partial k} \right)_{k=k^*} \).

(2.28) implies that around the steady state \( k^* \), \( k \) approaches \( k^* \) with the speed that depends on its difference from \( k^* \). The rate at which \( k(t) - k^* \) is reduced is approximately constant and equal to \( \lambda \). We shall refer to \( \lambda \) as the \textit{speed of convergence}. 

\[
\frac{s}{y^*} \frac{\partial y^*}{\partial s} = \frac{k^* f'(k^*) / f(k^*)}{1 - [k^* f'(k^*) / f(k^*)]} = \frac{\alpha_k(k^*)}{1 - \alpha_k(k^*)} \tag{2.25}
\]
Solving the first order linear differential equation (2.28), we get that,

\[ k(t) = k^* + e^{-\lambda t} (k(0) - k^*) \]  \hspace{1cm} (2.29)

where \( k(0) \) is the initial value of \( k \).

In order to calculate the speed of convergence \( \lambda \) in terms of the structural parameters of the model, we take the first derivative of the original non-linear differential equation (2.26) with respect to \( k \).

\[
\lambda = - \left( \frac{\partial k^*}{\partial k} \right)_{k=k^*} = - \left[ sf'(k^*) - (n+g+\delta) \right] = (n+g+\delta) - sf'(k^*)
\]

\[
= (n+g+\delta) \left[ 1 - k^* f'(k^*)/f(k^*) \right]
\]

\[
= (n+g+\delta) \left[ 1 - \alpha_K(k^*) \right] \hspace{1cm} (2.30)
\]

where \( \alpha_K(k^*) \) is the share of capital in total income at the steady state. In order to get to the final expression in (2.30) we used the fact that in the steady state \( sf(k^*) = (n+g+\delta)k^* \) in order to eliminate \( s \).

Widely accepted annual estimates of \( n+g+\delta \) determine it at about 6%. For example, this would be the result with \( n = 1\% \), \( g = 2\% \) and \( \delta = 3\% \). With the share of capital estimated at about 1/3, (2.30) implies an annual speed of convergence of about 4%.

Thus, on the basis of these widely accepted estimates, the Solow model implies that each year roughly 4% of the difference between the current capital stock (and income) and the steady state capital stock (and income) is covered through the process of capital accumulation.

From (2.29) we can estimate how many years it will take with this speed of convergence to cover a particular percentage of the gap between \( k(0) \) and \( k^* \).

In order to calculate the number of years required to cover half of the initial difference, we need to calculate the time span \( t \) that satisfies,

\[ e^{-\lambda t} = 0.5 \]

for \( \lambda = 4\% \). This suggests that \( t = - \ln(0.5)/\lambda = 0.69/\lambda = 0.69/0.04 = 17.3 \).

It would take 17.3 years to cover half of any initial difference between the capital stock (and real income) and its steady state value. This is often referred to as the half life of the convergence process.

In order to calculate the number of years required to cover two thirds of the initial difference, we need to calculate the time span \( t \) that satisfies,

\[ e^{-\lambda t} = 0.333 \]
for $\lambda=4\%$. This suggests that $t = -\ln(0.333)/\lambda = 2.1/\lambda = 2.1/0.04 = 27.5$.

It would take 27.5 years to cover two thirds of any initial difference between the capital stock (and real income) and its steady state value.

Econometric evidence from, among others, Mankiw, Romer and Weil (1992), suggests that the speed of convergence in the post war period was on average about 2% per annum. Thus, the speed of convergence predicted by the Solow model, based on the parameter estimates we used, is on the high side compared with the econometric evidence. We shall return to this issue in Chapter 6.

### 2.4 Competitive Markets, the Real Interest Rate and Real Wages

As we have presented it so far, the Solow model assumes a single domestic firm and one national household which owns this firm. However, due to the constant returns to scale hypothesis, all properties of this model go through, when one assumes competitive markets, with many firms and many households.

Suppose there is a large number of households owning capital and supplying one unit of labor per member. The interest rate is $r(t)$ and the real wage (per efficiency unit of labor) is $w(t)$. Each firm uses capital and labor and produces according to a production function which, in intensive form, is given by (2.2). Each firm pays the return on capital to households holding its shares, and real wages to its workers.

The conditions for profit maximization on the part of firms are that the marginal product of capital equals the user cost of capital (the real interest rate plus the depreciation rate), and that the marginal product of labor equals the real wage. Therefore it holds that,

$$f'(k(t)) = r(t) + \delta \quad (2.31)$$

$$f(k(t)) - k(t)f''(k(t)) = w(t) \quad (2.32)$$

One can easily conclude that, when (2.31) and (2.32), are satisfied, firms have zero profits and factor payments exhaust real output.

The total household income per efficiency unit of labor is equal to gross output and is given by,

$$(r(t) + \delta)k(t) + w(t)$$

The condition equating savings and investment per efficiency unit of labor is given by,

$$\dot{k}(t) = s((r(t) + \delta)k(t) + w(t)) - (n + g + \delta)k(t) \quad (2.33)$$

Substituting (2.31) and (2.32) in (2.33), we have the basic accumulation equation of the Solow model.
\[ k(t) = sf(k(t)) - (n + g + \delta)k(t) \]

Consequently, the Solow model, as analyzed so far, is compatible with the existence of competitive markets for goods, labor and capital.

It is worth noting that since the real interest rate is equal to the marginal product of capital minus the depreciation rate, at the golden rule, the real interest rate must be equal to the long run growth rate \( g + n \). Thus, an alternative way to define the golden rule in a competitive economy, is to define it as the balanced growth path along which the real interest rate is equal to the long-run growth rate.

In the process of adjustment towards the balanced growth path from the left, i.e when the initial capital per efficiency unit of labor is less than its steady state value, real wages are rising and real interest rates are falling, reflecting the evolution of the marginal product of capital and the marginal product of labor.

On the balanced growth path the real wage (per efficiency unit of labor) remains constant and the same happens with the real interest rate. However, the real wage per employee, along with all other per capita figures, is growing at a rate \( g \), the exogenous growth rate of labor efficiency.

2.5 The Process of Economic Growth and the Solow Model

The Solow model, like any other economic model, is based on relatively simple and, many would claim, largely unrealistic assumptions. However, it constitutes a significant improvement over previous models which did not rely on the neoclassical production function. Such were for example the models of Harrod (1939) and Domar (1946), which were based on Leontieff (1941) production functions with constant coefficients.

The question is whether the model of Solow (and all models based on similar assumptions about the technology of production) can account in a satisfactory manner for the key features of the process of economic growth in the real world. To answer this question, we must return to what these main features are.

2.5.1 The Kaldor Stylized Facts of Economic Growth

An important first codification of the main empirical features pertaining to long run growth, is due to Kaldor (1961), who based them on the long run growth experience of Great Britain and the USA. According to Kaldor, a growth theory should be consistent with the following six (6) stylized facts about long-run growth:

2. Per capita GDP is growing over time, and the growth rate is not declining.
3. Physical capital per worker is growing over time.
4. The long run rate of return on capital is roughly constant.
5. The long run capital-output ratio is roughly constant.
6. The shares of labor and capital in the Gross Domestic Product do not display a long-term trend.
7. The growth rate of labor productivity varies substantially between countries.
These stylized facts remain in force today, with the addition of some newer ones.\(^4\)

The Solow model is at a first reading consistent with all of these basic empirical characteristics. However, the process of physical capital accumulation, which is the main engine of economic growth in the Solow model, is not sufficient as an explanation of either the long-run growth of output per worker that has been observed historically in almost all developed economies of the world, or the large differences in output per worker between developed and less developed economies.

Only a small part of these phenomena can be explained by the accumulation of physical capital. The largest part appears to be due to technical progress and to differences in total factor productivity and the efficiency of labor, which for the Solow model are considered exogenous.\(^5\)

The Solow model identifies three sources of differences in output per worker between countries or between periods: First, differences in capital per worker, secondly, differences in total factor productivity and labor efficiency, and thirdly, differences in initial conditions.

To analyze the impact of each of these differences, we will use the Solow model, assuming a Cobb-Douglas production function.

### 2.5.2 Differences in Economic Growth between Economies

In the Solow model, based on the Cobb-Douglas production function, capital per efficiency unit of labor on the balanced growth path is defined by the condition,

\[
sA(k^*)^\alpha = (n + g + \delta)k^*
\]  

(2.34)

From (2.34) if follows that,

\[
k^* = \left( \frac{sA}{n + g + \delta} \right)^\frac{1}{1-\alpha}
\]  

(2.35)

From (2.35), output per efficiency unit of labor is given by,

\[
y^* = A(k^*)^\alpha = A\left( \frac{sA}{n + g + \delta} \right)^\frac{\alpha}{1-\alpha}
\]  

(2.36)

The per capita product on the balanced growth path is given by,

---

\(^4\) Jones and Romer (2010) have recently codified a number of additional stylized facts that a theory of economic growth must be able to account for. We shall examine these additional stylized facts in Chapter 6.

\(^5\) It is worth mentioning that Kaldor, who was very critical of neoclassical theory, considered the Solow model to be incompatible with at least some of the stylized facts that he identified, mainly stylized facts 1, 2 and 6. The reason he was critical is that the Solow model is compatible with these facts only when one assumes exogenous technological progress, which drives the efficiency of labor, per capita income, per capita consumption and real wages along the balanced growth path.
where the hat (^) over a variable denotes the per capita magnitude.

Based on (2.37), differences in capital per worker, for realistic estimates of the parameters of the model, cannot explain the differences in output per worker that we observe in the real world.

For example, let us assume that we want to explain a ratio $x$ in output per worker between two economies, 1 (a developed economy) and 2 (a less developed economy). From (2.37), assuming that all other parameters except for the capital stock are the same between the two economies, we must have that

$y_1^*(t) = \frac{Y_1^*(t)}{L(t)} = y^* h(t) = A \left( k_1^* \right)^{\alpha} h(0) e^{\alpha'}$

$y_2^*(t) = \frac{Y_2^*(t)}{L(t)} = y^* h(t) = A \left( k_2^* \right)^{\alpha} h(0) e^{\alpha'}$

To explain this ratio, capital per worker should differ by $x$ to the power $1/\alpha$, where $\alpha$ is the share of capital in domestic income. Since $\alpha$ is of the order of 1/3, to explain that GDP per worker is currently in developed countries 17 times higher than in less developed countries, capital per worker should be 4913 times (17 raised to the 3rd power) higher. But this is not the case. In developed economies capital per worker is only 20-30 times higher than in less developed economies. Thus, we cannot account for differences in per capita output and income on the basis of differences in the per capita capital stock.

We can certify this indirectly as well. If the differences in output per worker were due only to differences in physical capital per worker, then we should observe huge differences in the rate of return to capital between periods and between countries. However, such huge differences do not exist.

To explain the large differences between developed and underdeveloped countries on the balanced growth path, we should allow for differences in total factor productivity and the efficiency of labor. Allowing for such differences in (2.38), we have that

$\frac{y_1^*(t)}{y_2^*(t)} = \frac{A_1 k_1(t)^{\alpha} \left(h_1(0) e^{\alpha'}\right)^{1-\alpha}}{A_2 k_2(t)^{\alpha} \left(h_2(0) e^{\alpha'}\right)^{1-\alpha}} = \frac{A_1 h_1(0) k_1(t)^{\alpha} \left(h_1(0) e^{\alpha'}\right)^{1-\alpha}}{A_2 h_2(0) k_2(t)^{\alpha} \left(h_2(0) e^{\alpha'}\right)^{1-\alpha}} = \frac{A_1 h_1(0)}{A_2 h_2(0)}$, where $h(t) = \frac{h(t)}{L(t)}$

Differences in total factor productivity and the initial efficiency of labor, along with differences in physical capital per worker, can explain almost all differences in output per worker that we observe in the real world. For example, if the developed countries have capital per worker 30 times higher than the less developed countries, a total factor productivity which is three times that of the less developed countries ($A_1 = 3A_2$) and three times the initial efficiency of labor ($h_1(0) = 3h_2(0)$), then (2.39) predicts that, along the balanced growth path, their output per worker and their per capita income will be about 17 times higher than those of the less developed countries.
However, total factor productivity and the efficiency of labor are not explained by the Solow model, but considered exogenous. Therefore, one could say that this model does not explain the process of long run growth, but only assumes it.\(^6\)

That is why this model, like other models based on similar assumptions about the technology of production, are classified as **exogenous growth models**. They assume that total factor productivity \(A\), the initial efficiency of labor \(h(0)\), and the rate of technological progress \(g\), are all exogenous parameters.

### 2.5.3 Conditional Convergence

Our analysis in the previous section makes clear that the process of convergence predicted by the Solow model does not entail convergence to the same per capita income for all economies. The per capita income to which an economy converges is determined by (2.36) and (2.37) as,

\[
y^*(t) = A(k^*)^\alpha h(0)e^{\theta t} = A\left(\frac{sA}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}} h(0)e^{\theta t} \quad \text{(2.40)}
\]

To the extent that parameters such as the rate of savings and investment \(s\), total factor productivity \(A\), the population growth rate \(n\), the depreciation rate \(\delta\) and the initial labor efficiency \(h(0)\) differ between two economies, these economies will converge towards different levels of per capita income, even if along the balanced growth path, per capita income is growing at the same rate of technological progress \(g\).

Convergence towards different levels of per capita income, which depend on the parameters characterizing the structure of different economies, is called **conditional convergence**. The per capita income towards which economies converge in the Solow model, and the other exogenous growth models which we will analyze in the next few chapters, depends on their specific characteristics. Not all economies converge to the same per capita income. Each economy converges to the per capita income which is determined by its own technological, demographic and savings (investment) parameters.

### 2.5.4 Convergence with a Cobb Douglas Production Function

For the Solow model with a Cobb Douglas production function, one can solve in detail not only for the various variables along the balanced growth path, as we have done so far, but also for their path along the convergence process. This is because the first order differential equation that characterizes the convergence process in this case has the form of a Bernoulli equation, which can be converted to a linear equation in the capital-output ratio, and thus solved analytically.

With a Cobb Douglas production function output per efficiency unit of labor is given by,

---

\(^6\) Mankiw, Romer and Weil (1992) have generalized the Solow model, attributing differences in the efficiency of labor to investment in human capital (education of the labor force). However, they retained the assumption that total factor productivity increases at an exogenous rate \(g\). The generalized Solow model which they put forward seems to explain the growth experience of 98 non-oil producing countries after 1960 fairly well. See also Jones (2002) and Chapter 6 for generalized models for economic growth which rely on investment in both physical and human capital.
Therefore, the adjustment of capital per efficiency unit of labor is given by,

\[ k(t) = sA(t)^\alpha - (n + g + \delta)k(t) \] (2.41)

This is a Bernoulli equation, which can be converted to a linear differential equation if we define a new variable \( z \), as,

\[ z(t) = \frac{k(t)}{y(t)} = \frac{1}{A} k(t)^{1-\alpha} \] (2.42)

This variable is none other than the capital-output ratio.

From (2.42) it follows that,

\[ z(t) = \frac{\partial z(t)}{\partial k(t)} \cdot k(t) = \frac{1-\alpha}{A} k(t)^{-\alpha} \cdot k(t) \] (2.43)

By substituting (2.41) in (2.43) we get,

\[ z(t) = (1-\alpha)s - \lambda z(t) \] (2.44)

where \( \lambda = (1-\alpha)(n+g+\delta) \). The parameter \( \lambda \) is just the speed of convergence.

(2.44) is a first order linear differential equation in the variable \( z \) (the capital output ratio), and can be solved as,

\[ z(t) = \frac{s}{n+g+\delta} + \left( z_0 - \frac{s}{n+g+\delta} \right) e^{-\lambda t} \] (2.45)

where \( z_0 = \frac{1}{A} k_0^{1-\alpha} \) is the initial capital-output ratio.

Substituting from the definition of the capital-output ratio with a Cobb Douglas production function, the convergence process of capital and output per efficiency unit of labor is given by,

\[ k(t) = \left[ \frac{As}{n+g+\delta} \left( 1 - e^{-\lambda t} \right) + k_0^{1-\alpha} e^{-\lambda t} \right]^{\frac{1}{1-\alpha}} \] (2.46)

---

7 This solution method is proposed by Jones (2002).
The limit of (2.46) and (2.47), as time tends towards infinity, is the balanced growth path, as determined by (2.35) and (2.36).

2.6 Dynamic Simulations of the Solow Model

In order to investigate further the process of dynamic adjustment the characterizes the Solow model, we can simulate, for specific values of the parameters of the model, the transition from a balanced growth path to another, when there is an exogenous permanent change in specific parameters, such as the savings rate or total factor productivity.

To simulate the model numerically we shall convert it from a continuous-time model to a discrete-time model (see Annex to Chapter 1).

In discrete time, the accumulation equation of capital per efficiency unit of labor is given by,

\[ k_{t+1} = \frac{1}{(1+n)(1+g)} \left( sf(k_t) + (1-\delta)k_t \right) \]  

(2.48)

It can easily be shown diagrammatically (see. Figure 2.5), that the difference equation (2.48) converges to a unique equilibrium. The process of convergence is determined by (2.48) and the equilibrium towards which the economy converges determines the balanced growth path.

For the purposes of the simulation we shall assume that the production function is Cobb Douglas,

\[ y_t = f(k_t) = Ak_t^\alpha \]  

(2.49)

where \( A > 0 \) is total factor productivity, and \( 0 < \alpha < 1 \) the exponent (share) of capital in the production function. \( 1-\alpha \) is the exponent of labor.

Substituting (2.49) in (2.48), the capital accumulation equation is given by,

\[ k_{t+1} = \frac{1}{(1+n)(1+g)} \left( sf(k_t^*) + (1-\delta)k_t^* \right) \]  

(2.50)

From (2.50), the steady state capital stock, per efficiency unit of labor, is given by,

\[ k^* = \left( \frac{sA}{(1+n)(1+g)-(1-\delta)} \right)^{\frac{1}{1-\alpha}} \]  

(2.51)

The remaining variables are all functions of \( k \) and their steady state values a function of \( k^* \).

Output is given by (2.49), and steady state output is given by,
\[ y^* = A \left( \frac{sA}{(1+n)(1+g)-(1-\delta)} \right)^{\frac{\alpha}{\alpha-1}} \] (2.52)

Consumption is given by,

\[ c_t = (1-s)A^\alpha k_t^{\alpha-1} \] (2.53)

Finally, the real interest rate and the real wage are given by,

\[ r_t = \alpha A k_t^{\alpha-1} - \delta \] (2.54)

\[ w_t = (1-\alpha)A k_t^\alpha \] (2.55)

Solving (2.50) on the computer, for specific parameter values, we can calculate the dynamic adjustment of capital towards the balanced growth path. The dynamic adjustment of the other variables can be calculated then from (2.49), (2.53), (2.54) and (2.55).

In Figure 2.6 we present the dynamic adjustment of the Solow model following a permanent increase in the saving rate \( s \) by 5%. In Figure 2.7 we present the dynamic adjustment of the model following an increase in total factor productivity \( A \) by 5%.

The values of the initial parameters in the simulation are as follows: \( A=1, \alpha=0.333, s=0.30, n=0.01, g=0.02, \delta=0.03 \). These values correspond to the values used to calculate the speed of convergence in section 2.4.

In the simulation of Figure 2.6, the economy is on the initial balanced growth path, and in period 1, the savings rate increases permanently and unexpectedly by 5%, from 0.30 to 0.315. The increase in the saving rate leads to a decrease in consumption, gradual accumulation of capital, a gradual increase in production, a gradual increase in real wages and a gradual fall in real interest rates. The reason behind increasing real wages is the gradual increase in the marginal product of labor caused by the accumulation of capital. The reason behind the gradual reduction in the real interest rate is the gradual reduction of the marginal product of capital caused by the accumulation of capital. The economy gradually converges towards a new balanced growth path. In this new balanced growth path, capital per efficiency unit of labor is higher by approximately 7.6%, output and real wages by 2.5%, consumption by 0.3% (due to the increase in the saving rate), while the real interest rate has fallen by 0.3 percentage points.

In the simulation of Figure 2.7 the economy is on the initial balanced growth path, and in period 1, total factor productivity \( A \) increases permanently and unexpectedly by 5%, from 1 to 2.05. This increase leads immediately to an increase in output, consumption, savings, the marginal product of labor (real wage) and the marginal product of capital (real interest rate). The increase in savings causes gradual accumulation of capital, which leads to a further gradual increase in output and consumption, a further gradual increase in real wages, but a gradual fall in real interest rates. The reason for the gradual decrease of the real interest rate is the gradual reduction of the marginal product of capital caused by the accumulation of capital. The economy gradually converges to a new balanced growth path. In this, capital per efficiency unit of labor is increased by about 7.6%,
output, consumption and real wages also increased by 7.6%, while the real interest rate, after the initial increase, has returned to its original equilibrium. The equilibrium real interest rate, because the production function is assumed Cobb Douglas, is independent of total factor productivity $A$. The reason why an increase in productivity by 5% leads to an increase in real income by 7.6%, i.e. more than 5%, is that the increase in productivity causes an increase in savings and capital accumulation, which in turn causes additional secondary increases in real incomes and consumption. This can be confirmed from (2.52), where the elasticity of steady state output with respect to total factor productivity $A$ is equal to $1/(1-\alpha)>2$.

2.7 Conclusions

The Solow model is a key model in the theory of economic growth. Although it is rooted in older models, and although it has theoretical and empirical weaknesses, this model provides an extremely useful, relatively simple, and flexible framework for the analysis of the process of economic growth.

However, the process of physical capital accumulation, which is the main engine of economic growth in the Solow model, cannot fully explain either the long-run growth in output per worker that has been observed in developed economies, or the large differences in output per worker between developed and less developed economies. In fact, only a small part of these phenomena can be explained by the accumulation of physical capital. Much more is due to total factor productivity and the efficiency of labor (technological progress), which for the Solow model are considered exogenous parameters.

In this sense, the Solow model, and all the models that make similar assumptions about technology and technological progress, shows us how to overcome its weaknesses and to try to explain technological progress. This is the main difference of this model, and all exogenous growth models, from the endogenous growth models that we shall examine in Chapter 6.

Another theoretical weakness of the Solow model is the assumption that the savings rate is exogenous. Although at the time that the Solow model first appeared this was a widespread assumption in the context of Keynesian economics, the assumption is not satisfactory as it does not take into account the underlying determinants of household savings behavior. In the next two chapters we examine two alternative classes of models of savings behavior, where savings are the result of rational inter-temporal behavior on the part of households that have access to the capital market. These two classes of models, which are the basis of modern inter-temporal macroeconomics, are models of a representative household and models of overlapping generations.
Annex to Chapter 2
The Solow Model in Discrete Time

This Annex sets out the Solow model in discrete time. Instead of assuming that time is a continuous variable, time is now measured as successive time periods, where \( t = 0, 1, 2, \ldots \). The variable \( x_t \) indicates the variable \( x \) in period \( t \).

Population and the efficiency of labor grow at rates \( n \) and \( g \) per period respectively. Thus, we have,

\[
L_t = L_0(1 + n)^t \quad \text{(A2.1)}
\]

\[
h_t = h_0(1 + g)^t \quad \text{(A2.2)}
\]

The production function is given by,

\[
Y_t = F(K_t, h_t L_t) \quad \text{(A2.3)}
\]

and is characterized by constant returns to scale and diminishing returns of individual factors.

We assume, as in the case of continuous time, that the consumption function is characterized by a fixed savings rate \( s \).

\[
C_t = (1 - s)Y_t = (1 - s)F(K_t, h_t L_t) \quad \text{(A2.4)}
\]

The accumulation of capital is determined by,

\[
K_{t+1} - K_t = F(K_t, h_t L_t) - C_t - \delta K_t = sF(K_t, h_t L_t) - \delta K_t \quad \text{(A2.5)}
\]

With these assumptions, we can express all variables per efficiency unit of labor.

\[
y_t = f(k_t) \quad \text{(A2.6)}
\]

\[
c_t = (1 - s)y_t = (1 - s)f(k_t) \quad \text{(A2.7)}
\]

\[
k_{t+1} = \frac{1}{(1 + n)(1 + g)} \left( f(k_t) - c_t + (1 - \delta)k_t \right) \quad \text{(A2.8)}
\]

Substituting (A2.7) to (A2.8) we get the basic equation of capital accumulation in the Solow model in discrete time. This is a non-linear first-order difference equation and has the form,

\[
k_{t+1} = \frac{1}{(1 + n)(1 + g)} \left( sf(k_t) + (1 - \delta)k_t \right) \quad \text{(A2.9)}
\]

The equilibrium capital per efficiency unit of labor is determined by the relationship,
The dynamic adjustment towards equilibrium through the difference equation (A2.9) is depicted in Figure 2.5. The equilibrium is unique and stable, and the economy converges to it from any initial point.

\[ sf(k^*) = (n + g + \delta + ng)k^* \]
Figure 2.1
The Production Function in Intensive Form
Figure 2.2
Equilibrium in the Solow Model

\[ y^* = \frac{(n + g + \delta)k}{s} \]

Graph showing the equilibrium in the Solow Model with various curves and points labeled.
Figure 2.3
Implications of a Rise in the Savings Rate

\[ y, s \]

\[ y^{**} \]

\[ y^{*} \]

\[ (n + g + \delta)k \]

\[ s'f(k) \]

\[ sf(k) \]

\[ f(k) \]

\[ k^{*} \]

\[ k^{**} \]
Figure 2.4
The Process of Convergence to a New Balanced Growth Path

The graph shows the process of convergence to a new balanced growth path. The variables $k$ (capital per worker) and $t$ (time) are plotted on the vertical and horizontal axes, respectively. The initial level of capital per worker is $k^*$, and the target level is $k^{**}$. The curve illustrates the path of capital per worker over time, converging towards $k^{**}$. The process is represented by a smooth curve that starts from $k^*$ and approaches $k^{**}$ as time progresses.
Equilibrium in the Solow Model in Discrete Time

\[ k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{(1+n)(1+g)} \]

Figure 2.5
Figure 2.6
Dynamic Adjustment of the Solow Model
Following a Permanent Increase in the Savings Rate by 5%
Figure 2.7
Dynamic Adjustment of the Solow Model
Following a Permanent Increase in Total Factor Productivity by 5%
References